Announcements

1) HW #3 - one webwork question and written homework due Thursday next week.

Example 1 : Solve

$$2y'' - 8y = 9e^{3t} \text{ for } y.$$

Two solutions differ by a solution to

$2y'' - 8y = 0$, first solve the homogeneous equation.

Suppose $y = e^{rt}$. Then we get

$$2r^2e^{rt} - 8e^{rt} = 0,$$

so $2r^2 - 8 = 0$

$$r^2 = 4$$

$$r = \pm 2.$$

Solutions to Homogeneous Equation: $\boxed{C_1 e^{2t} + C_2 e^{-2t}}$

We just need one solution to

$$2y'' - 8y = 9e^{3t};$$

any other solution will be obtained by adding $C_1 e^{2t} + C_2 e^{-2t}$ for some real numbers C_1, C_2.

We wish:

that a solution to the problem looks like

$$\boxed{y = u(t)e^{2t} + v(t)e^{-2t}}$$

for functions u and v.

Then

$$y' = \left(u(t)e^{2t} + v(t)e^{-2t}\right)'$$

$$= u'(t)e^{2t} + 2u(t)e^{2t}$$
$$+ v'(t)e^{-2t} - 2v(t)e^{-2t}$$
$$= 2\left(u(t)e^{2t} - v(t)e^{-2t}\right)$$
$$+ \underbrace{\left(v'(t)e^{-2t} + u'(t)e^{2t}\right)}_{\text{I wish this to be zero!}}$$

If $v'(t)e^{-2t} + u'(t)e^{2t} = 0$,

then

$$y' = 2\left(u(t)e^{2t} - v(t)e^{-2t}\right),$$

so

$$y'' = 2\Big(u'(t)e^{2t} + 2u(t)e^{2t} - v'(t)e^{-2t} + 2v(t)e^{-2t}\Big)$$

Plug back into original equation.

$$9e^{3t} = 2y'' - 8y$$

$$= 4\left(u'(t)e^{2t} + 2u(t)e^{2t} - v'(t)e^{-2t} + 2v(t)e^{-2t}\right)$$

$$-8\left(u(t)e^{2t} + v(t)e^{-2t}\right)$$

$$= 4u'(t)e^{2t} + \cancel{8u(t)e^{2t}}$$

$$-4v'(t)e^{-2t} + \cancel{8v(t)e^{-2t}}$$

$$-\cancel{8u(t)e^{2t}} - \cancel{8v(t)e^{-2t}}$$

We now have two equations:

$$1)\quad u'(t)e^{2t} + v'(t)e^{-2t} = 0$$

$$2)\quad 4u'(t)e^{2t} - 4v'(t)e^{-2t} = 9e^{3t}$$

$$u'(t)e^{2t} - v'(t)e^{-2t} = \frac{9}{4}e^{3t}$$

Add the two equations to get

$$2u'(t)e^{2t} = \frac{9}{4}e^{3t},\ \text{so}$$

$$u'(t) = \frac{9}{8}e^{t}$$

Subtracting 2) from 1),

we get

$$2v'(t)e^{-2t} = -\frac{9}{4}e^{3t}$$

$$v'(t) = -\frac{9}{8}e^{5t}.$$

We have: $u'(t) = \frac{9}{8}e^{t}$

$v'(t) = -\frac{9}{8}e^{5t}$

Integrating, we get

$u(t) = \frac{9}{8}e^{t}$, $v(t) = \frac{-9}{40}e^{5t}$.

A solution to $2y'' - 8y = 9e^{3t}$

is given by

$$y_p = \left(\frac{9}{8}e^{t}\right)\left(e^{2t}\right) + \left(\frac{-9}{40}e^{5t}\right)\left(e^{-2t}\right)$$

Simplifying,

$$y_p = e^{3t}\left(\frac{9}{8} - \frac{9}{40}\right)$$

$$= e^{3t}\left(\frac{9}{10}\right) = \boxed{\frac{9e^{3t}}{10}}$$

(subscript "p" stands for "particular"). Then a general solution is given by

$$\boxed{y = \frac{9}{10}e^{3t} + C_1 e^{2t} + C_2 e^{-2t}}$$

This trick is called

Variation of Parameters

and it works for any nonhomogeneous, second order differential equation, as follows:

1) Find solutions to the homogeneous equation.

2) Given solutions h and g to the homogeneous system, assume that a solution y_p to the nonhomogeneous system looks like

$$\boxed{y_p = u(t)h(t) + v(t)g(t)}$$

for some functions u, v with

$$\boxed{u'(t)h(t) + v'(t)g(t) = 0}$$

3) If the original equation is $ay'' + by' + cy = f$,

plug y_p into this equation. It will reduce to

$$\boxed{u'(t)h'(t) + v'(t)g'(t) = \frac{f(t)}{a}}$$

4) We now have two equations

$$\boxed{\begin{aligned} u'(t)h(t) + v'(t)g(t) &= 0 \\ u'(t)h'(t) + v'(t)g'(t) &= \frac{f(t)}{a} \end{aligned}}$$

Multiply the first equation by

$-\dfrac{g'(t)}{g(t)}$, add to the second.

We get

$$u'(t)\left(-\frac{h(t)g'(t)}{g(t)}+h'(t)\right)=\frac{f(t)}{a}.$$

With a common denominator,

$$u'(t)\left(\frac{h'(t)g(t)-h(t)g'(t)}{g(t)}\right)=\frac{f(t)}{a}$$

So

$$\boxed{u'(t)=\frac{1}{a}\,\frac{f(t)g(t)}{h'(t)g(t)-h(t)g'(t)}}$$

Integrate and solve for u.

Similarly,

$$\boxed{v'(t) = \frac{1}{a} \frac{f(t)h(t)}{h(t)g'(t) - h'(t)g(t)}}$$

Integrate and solve for v.

Final solution:

$$\boxed{y = y_p + C_1 h + C_2 g}$$